1. Positive eigenvectors for positive matrices

Let V be a finite or countably infinite set.

$$A : V \times V \to [0, \infty) \quad A(v, w) = A_{vw}.$$

Basic question: For which $\beta \in \mathbb{R}$ is there a non-zero vector $\psi_v, \ v \in V$, such that $\psi_v \geq 0$ and

$$\sum_{w \in W} A_{vw} \psi_w = e^{\beta} \psi_v \quad \forall v \in V \quad ?$$

2. Irreducibility

Define $A^n, n \in \mathbb{N}$, recursively such that

$$A^0_{vw} = \begin{cases} 0, & v \neq w \\ 1, & v = w \end{cases}$$

and

$$A^n_{vw} = \sum_{u \in V} A_{vu} A^{n-1}_{uw}, \ n \geq 1.$$

Note A^n may have ∞ among its entries.

We assume in the following thaty A is *irreducible*, meaning that

$$\forall v, w \in V \ \exists n \in \mathbb{N} : \ A^n_{vw} > 0$$

3. Graphs

A defines in a canonical way an oriented graph with V as the set of vertexes.

$$v \overset{A_{vw}}{\to} w$$

A is irreducible when this graph is strongly connected: For any pair of vertexes v, w there is a path from v to w.

4. Necessary conditions

Assume that there is a positive e^{β}-eigenvector for A. Then

$$A^n_{vw} < \infty \ \forall n, v, w,$$

and

$$\log \left(\limsup_n \left(A^n_{vv}\right)^{\frac{1}{n}} \right) \leq \beta \tag{4.1}$$ e2

Proof. Since

$$\sum_{w \in V} A^n_{vw} \psi_w = e^{n\beta} \psi_v, \tag{4.2}$$ e1

we see that $\psi_v > 0$ for all $v \in V$. It follows then from (4.2) that $A^n_{vw} < \infty$ for all n, v, w.

Note that

$$A^n_{vv} \psi_v \leq \sum_{w \in V} A^n_{vw} \psi_w = e^{n\beta} \psi_v,$$

which implies that

$$\left(A^n_{vv}\right)^{\frac{1}{n}} \psi_v^{\frac{1}{n}} \leq e^{\beta} \psi_v^{\frac{1}{n}}.$$

This implies (4.1).

We assume in the following that $A^n_{vw} < \infty$ for all n, v, w and that

$$\log\left(\limsup_n \left(A^n_{vv}\right)^{\frac{1}{n}}\right) < \infty.$$

5. When V is finite

From (4.2) we see that

$$\sum_{w \in V} \sum_{n=0}^{\infty} A^n_{vw} e^{-n\beta} \psi_w = \sum_{n=0}^{\infty} \psi_v = \infty.$$

If V is finite we conclude that

$$\sum_{n=0}^{\infty} A^n_{vw} e^{-n\beta} = \infty$$

for some $w \in V$. Choose $k \in \mathbb{N}$ such that $A^k_{wv} > 0$. Then

$$\begin{aligned}
\infty &= A^k_{wv} \sum_{n=0}^{\infty} A^n_{vw} e^{-n\beta} \\
&= \sum_{n=0}^{\infty} A^n_{vw} A^k_{wv} e^{-n\beta} \leq \sum_{n=0}^{\infty} A^{n+k}_{vv} e^{-n\beta} \\
&= e^{k\beta} \sum_{n=0}^{\infty} A^{n+k}_{vv} e^{-(n+k)\beta} \leq e^{k\beta} \sum_{n=0}^{\infty} A^n_{vv} e^{-n\beta}.
\end{aligned}$$

Hence $\sum_{n=0}^{\infty} A^n_{vv} e^{-n\beta} = \infty$ which implies that

$$e^{-\beta} \geq \frac{1}{\limsup_n \left(A^n_{vv}\right)^{\frac{1}{n}}},$$

or

$$\beta \leq \log\left(\limsup_n \left(A^n_{vv}\right)^{\frac{1}{n}}\right).$$

So when V is finite the basic question only has a positive answer when

$$\beta = \log\left(\limsup_n \left(A^n_{vv}\right)^{\frac{1}{n}}\right).$$

□

6. Existence of the solution when $\sum_{n=0}^{\infty} A^n_{vv} e^{-n\beta} = \infty$.

Set $\beta_0 = \log\left(\limsup_n \left(A^n_{vv}\right)^{\frac{1}{n}}\right)$.

Introduce the numbers $r_{vw}(n)$ such that $r_{vw}(0) = 0$, $r_{vw}(1) = A_{vw}$ and

$$r_{vw}(n+1) = \sum_{u \neq w} A_{vu} r_{uw}(n)$$

when $n \geq 1$. Then

$$\sum_{n=0}^{\infty} A^n_{vw} e^{-n\beta} = I_{vw} + \left(\sum_{n=1}^{\infty} r_{vw}(n) e^{-n\beta}\right)\left(\sum_{n=0}^{\infty} A^n_{ww} e^{-n\beta}\right). \tag{6.1}$$

e3

for all $v, w \in V$ when $\beta > \beta_0$. This follows from the product rule for power series by use of the observation that for $n \geq 1$,

$$A^n_{vw} = \sum_{s=1}^{n} r_{vw}(s) A^{n-s}_{ww}.$$

It follows that

$$\left(\sum_{n=1}^{\infty} r_{vv}(n) e^{-n\beta} \right) < 1$$

when $\beta > \beta_0$, and since

$$\sum_{n=0}^{\infty} A^n_{vv} e^{-n\beta} = \frac{1}{1 - \sum_{n=1}^{\infty} r_{vv}(n) e^{-n\beta}},$$

that

$$\sum_{n=1}^{\infty} r_{vv}(n) e^{-n\beta_0} = 1 \tag{6.2}$$

e5

when $\sum_{n=0}^{\infty} A^n_{vv}(n) e^{-n\beta_0} = \infty$.

Now note that

$$\begin{aligned}
&\sum_{u \in V} A_{vu} \left(\sum_{n=1}^{N} r_{uw}(n) e^{-n\beta_0} \right) \\
&= \sum_{n=1}^{N} \sum_{u \neq w} A_{vu} r_{uw}(n) e^{-n\beta_0} + A_{vw} \sum_{n=1}^{N} r_{ww}(n) e^{-n\beta_0} \\
&= \sum_{n=1}^{N} r_{vw}(n+1) e^{-n\beta_0} + A_{vw} \sum_{n=1}^{N} r_{ww}(n) e^{-n\beta_0} \qquad (6.3) \\
&= e^{\beta_0} \sum_{n=1}^{N} r_{vw}(n+1) e^{-(n+1)\beta_0} + A_{vw} \sum_{n=1}^{N} r_{ww}(n) e^{-n\beta_0} \\
&= e^{\beta_0} \sum_{n=1}^{N+1} r_{vw}(n) e^{-n\beta_0} + A_{vw} \left(\sum_{n=1}^{N} r_{ww}(n) e^{-n\beta_0} - 1 \right).
\end{aligned}$$

i10

Let $N \to \infty$ and use (6.2) to find that

$$\psi_v = \sum_{n=1}^{\infty} r_{vw}(n) e^{-n\beta_0}$$

is a positive e^{β_0}-eigenvector for A.

7. Uniqueness of the positive eigenvector when $\sum_{n=0}^{\infty} A^n_{vv} e^{-n\beta} = \infty$.

Let $\xi = (\xi_v)_{v \in V} \in [0, \infty)^V$ be a solution to (4.2) such that

$$\xi_{v_0} = 1.$$

We prove by induction that

$$\sum_{n=1}^{N} r_{vv_0}(n) e^{-n\beta} \leq \xi_v \tag{7.1}$$

b12

for all N and all v. To start the induction note that $\xi_v = e^{-\beta}\sum_{w\in V} A_{vw}\xi_w \geq e^{-\beta}A_{vv_0}\xi_{v_0} = r_{vv_0}(1)e^{-\beta}$. Assume then that (7.1) holds for all v. It follows that

$$\begin{aligned}
\xi_v &= e^{-\beta}\sum_{w\in V} A_{vw}\xi_w = e^{-\beta}\left(\sum_{w\neq v_0} A_{vw}\xi_w + A_{vv_0}\right) \\
&\geq e^{-\beta}\sum_{n=1}^{N}\sum_{w\neq v_0} A_{vw}r_{wv_0}(n)e^{-n\beta} + e^{-\beta}A_{vv_0} \\
&= \sum_{n=1}^{N} r_{vv_0}(n+1)e^{-(n+1)\beta} + e^{-\beta}r_{vv_0}(1) = \sum_{n=1}^{N+1} r_{vv_0}(n)e^{-n\beta}
\end{aligned}$$

Hence (7.1) follows by induction and we conclude that

$$\xi_v \geq \sum_{n=1}^{\infty} r_{vv_0}(n)e^{-n\beta} := \psi_v \tag{7.2}$$

for all v. However,

$$e^{n\beta_0} = e^{n\beta_0}\psi_{v_0} = e^{n\beta_0}\xi_{v_0} = \sum_{w\in V} A^n_{v_0w}\psi_w = \sum_{w\in V} A^n_{v_0w}\xi_w \tag{7.3}$$

for all $n \in \mathbb{N}$. If $\psi_v \neq \xi_v$ for just a single $v \in V$, we could use the irreducibility of A to choose $n \in \mathbb{N}$ such that

$$A^n_{v_0v}\psi_v > A^n_{v_0v}\xi_v.$$

Thanks to (7.2) this would contradict (7.3). It follows that

$$\sum_{n=1}^{\infty} r_{vv_0}(n)e^{-n\beta}, \ v \in V,$$

is the only positive e^{β_0}-eigenvector for A, up to multiplication by scalars.

8. Eigenvectors when $\beta > \beta_0$

When $\beta > \beta_0$ the sums $\sum_{n=0}^{\infty} A^n_{vw}e^{-n\beta}$ are finite. Sometimes this also true when $\beta = \beta_0$. We consider now the case where this is finite for all v, w. Fix a vertex $v_0 \in V$, and consider any other $v \in V$. There is then an $m \in \mathbb{N}$ such that $A^m_{v_0v} > 0$. It follows that

$$\begin{aligned}
A^m_{v_0v}\sum_{n=0}^{\infty} A^n_{vw}e^{-n\beta} &\leq \sum_{n=0}^{\infty} A^{m+n}_{v_0w}e^{-n\beta} \\
&= e^{m\beta}\sum_{n=0}^{\infty} A^{n+m}_{v_0w}e^{(-m-n)\beta} \leq e^{m\beta}\sum_{n=0}^{\infty} A^n_{v_0w}e^{-n\beta}
\end{aligned} \tag{8.1}$$

and hence

$$\frac{\sum_{n=0}^{\infty} A^n_{vw}e^{-n\beta}}{\sum_{n=0}^{\infty} A^n_{v_0w}e^{-n\beta}} \leq \frac{e^{m\beta}}{A^m_{v_0v}} \tag{8.2}$$

Let $\{w_k\}$ be a sequence of vertexes such that

$$\forall v \in V \ \exists N \in \mathbb{N} : \ w_k \neq v \ \forall k \geq N.$$

Since

$$\frac{\sum_{n=0}^{\infty} A^n_{vw_k} e^{-n\beta}}{\sum_{n=0}^{\infty} A^n_{v_0 w_k} e^{-n\beta}} \leq \frac{e^{m\beta}}{A^m_{v_0 v}}$$

for every $v \in V$, and V is countable there is a (diagonal) subsequence $\{w_{k_i}\}$ such that

$$\psi_v = \lim_{i \to \infty} \frac{\sum_{n=0}^{\infty} A^n_{vw_{k_i}} e^{-n\beta}}{\sum_{n=0}^{\infty} A^n_{v_0 w_{k_i}} e^{-n\beta}}$$

exists for all $v \in V$. Note that

$$\sum_{u \in V} A_{vu} \sum_{n=0}^{N} A^n_{uw_{k_i}} e^{-n\beta} = e^{\beta} \sum_{n=0}^{N+1} A^n_{vw_{k_i}} e^{-n\beta} - e^{\beta} I_{vw_{k_i}}, \tag{8.3}$$ i2

for all N, leading to the identity

$$\sum_{u \in V} A_{vu} \frac{\sum_{n=0}^{\infty} A^n_{uw_{k_i}} e^{-n\beta}}{\sum_{n=0}^{\infty} A^n_{v_0 w_{k_i}} e^{-n\beta}} = e^{\beta} \frac{\sum_{n=0}^{\infty} A^n_{vw_{k_i}} e^{-n\beta}}{\sum_{n=0}^{\infty} A^n_{v_0 w_{k_i}} e^{-n\beta}} - \frac{e^{\beta} I_{vw_{k_i}}}{\sum_{n=0}^{\infty} A^n_{v_0 w_{k_i}} e^{-n\beta}}.$$

If we boldly interchange summation and limit we get

$$\lim_{i \to \infty} \sum_{v \in V} A_{vu} \frac{\sum_{n=0}^{\infty} A^n_{uw_{k_i}} e^{-n\beta}}{\sum_{n=0}^{\infty} A^n_{v_0 w_{k_i}} e^{-n\beta}} = \sum_{u \in V} A_{vu} \psi_u. \tag{8.4}$$ boldly

Note that

$$\lim_{i \to \infty} \frac{e^{\beta} I_{vw_{k_i}}}{\sum_{n=0}^{\infty} A^n_{v_0 w_{k_i}} e^{-n\beta}} = 0$$

since $\lim_{i \to \infty} w_{k_i} = \infty$, and we can then conclude from (8.3) that

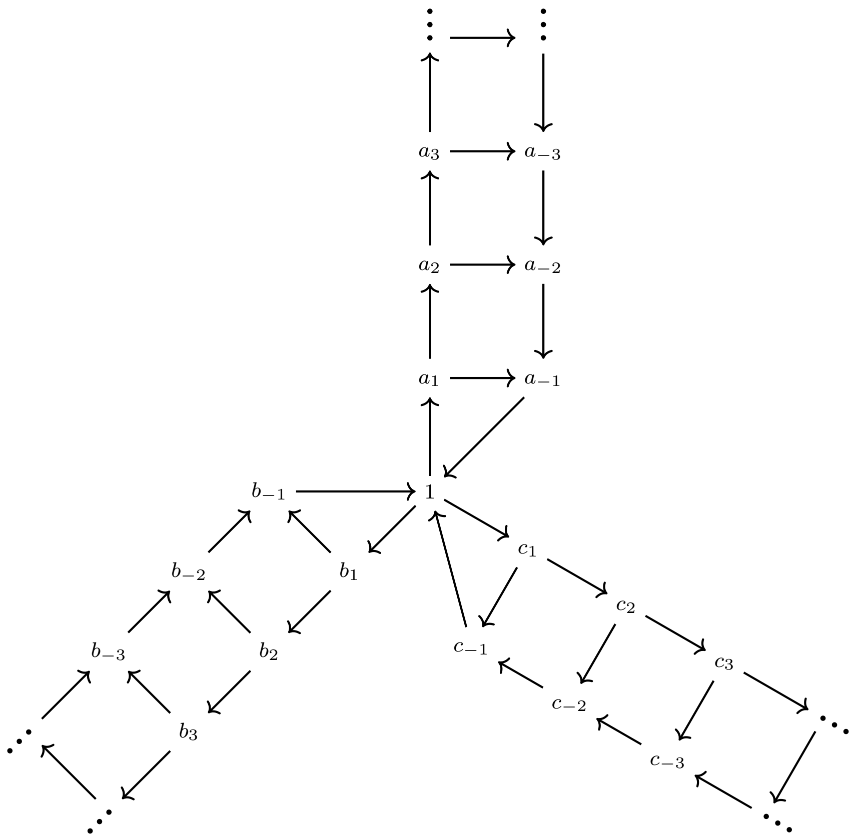
$$\sum_{u \in V} A_{vu} \psi_u = e^{\beta} \psi_v$$

for all $v \in V$. Since $\psi_{v_0} = 1$ we have obtained a solution to (4.2). The questionable step (8.4) is legitimate when A is row-finite, in the sense that

$$\# \{ w \in V : \ A_{vw} \neq 0 \} < \infty \ \forall v \in V.$$

Thus we have obtained a proof of a 1964 result of W.E. Pruitt: When A row-finite there is a positive e^{β}-eigenvector for all $\beta \geq \beta_0$.

example1 **Example 8.1.** Consider the following graph with labeled vertexes:



For this graph it is quite easy to see that a map $\xi : V \to [0, \infty)$ which is normalized such that $\xi_1 = 1$, is a positive e^β-eigenvector for the adjacency matrix A of the graph when

i) $\xi_{a_1} + \xi_{b_1} + \xi_{c_1} = e^\beta$,
ii) $\xi_{a_{-n}} = \xi_{b_{-n}} = \xi_{c_{-n}} = e^{-\beta n}, \ n = 1, 2, 3, \dots,$ and
iii) $\xi_{a_{n+1}} + e^{-n\beta} = e^\beta \xi_{a_n}, \ \xi_{b_{n+1}} + e^{-n\beta} = e^\beta \xi_{b_n}, \ \xi_{c_{n+1}} + e^{-n\beta} = e^\beta \xi_{c_n}, \ n \geq 1$

It follows that

$$\xi_{a_{n+1}} = e^{n\beta} \left(\xi_{a_1} - \sum_{j=1}^{n} \left(e^{-2\beta}\right)^j \right), \ n \geq 1,$$

combined with similar formulas involving the b_n's and c_n's. The positivity requirement on ξ implies that $\beta > 0$ and that

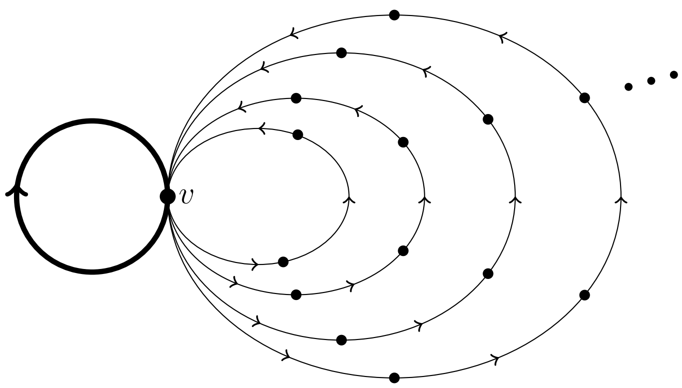
$$\min\{\xi_{a_1}, \xi_{b_1}, \xi_{c_1}\} \geq \sum_{j=1}^{\infty} (e^{-2\beta})^j = \frac{e^{-2\beta}}{1 - e^{-2\beta}}.$$

Combined with condition i) it follows that $3\frac{e^{-2\beta}}{1-e^{-2\beta}} \leq e^\beta$, which means that $\beta \geq \log \alpha \sim 0,5138$, where α is the real root of the polynomial $x^3 - x - 3$. For $\beta = \log \alpha$ there is a unique solution, and hence there is a unique positive e^β-eigenvector for A, up to scalar multiplication. For all values of $\beta > \log \alpha$ the set of β-KMS weights form a cone with a triangle as base. The extreme rays of the cone correspond to the three cases where

$$\{\xi_{a_1}, \xi_{b_1}, \xi_{c_1}\} = \left\{ \frac{e^{-2\beta}}{1 - e^{-2\beta}}, \ e^\beta - \frac{2e^{-2\beta}}{1 - e^{-2\beta}} \right\}.$$

When A is not row-finite, there is a problem with (8.4), and I aim to demonstrate by example that it is not only a technicality.

8.1. **Example.** Consider the following graph:



It is determined by a function $a : \mathbb{N} \to \mathbb{N}$ defined such that $a(n)$ is the number of loops of length n. For example we consider

$$a(n^2) = 2^{n^2-n}, \ n = 1, 2, 3, \cdots$$

and $a(k) = 0$ when k is not a square. Let V be the vertexes in the grpah and $A = (A_{vw})_{v,w \in V}$ the adjacency matrix of the graph, i.e.

$$A_{vw} = \text{number of edges from } v \text{ to } w.$$

If $\psi \in [0, \infty)^V$ is an e^β-eigenvector with $\psi_u = 1$, we must have that

$$e^\beta = 1 + \sum_{n=2}^{\infty} e^{-(n^2-1)\beta} 2^{n^2-n},$$

or

$$1 = e^{-\beta} + \sum_{n=2}^{\infty} e^{-n^2\beta} 2^{n^2-n}. \tag{8.5}$$ u20

For the sum to be convergent we must have that

$$\limsup_n -n^2\beta + (n^2 - n)\log 2 < 0,$$

which means that $\beta \geq \log 2$. Note that equality holds in (8.5) when $\beta = \log 2$. Since the righthand side is strictly decreasing in β, it follows that $\beta = \log 2$ is the only value for which there can be an e^β-eigenvector - and there is actually one, and it is unique (This is an exercise!).

Note that $\limsup_{n\to\infty} (A^n_{vv})^{\frac{1}{n}} = 2$. Indeed,

$$\limsup_{n\to\infty} (A^n_{vv})^{\frac{1}{n}} \geq \limsup_{n\to\infty} \left(2^{n^2-n}\right)^{\frac{1}{n^2}} = 2. \tag{8.6}$$ e10

On the other the presence of a positive 2-eigenvector implies that $2 \geq \limsup_{n\to\infty} (A^n_{vv})^{\frac{1}{n}}$, cf. (4.1). Hence A behaves as a finite matrix with respect to positive eigenvectors.

Now remove the edge from u to itself to get the graph G', and consider its adjacency matrix B. There are then no positive eigenvectors at all. Indeed, without the loop of length 1 at u, the equation u20 8.5 becomes

$$1 = \sum_{n=2}^{\infty} e^{-n^2\beta} 2^{n^2-n}. \qquad (8.7)$$ u22

Since the sum can only be convergent when $\beta \geq \log 2$ and

$$\sum_{n=2}^{\infty} e^{-n^2\beta} 2^{n^2-n} \leq \sum_{n=2}^{\infty} e^{-n^2 \log 2} 2^{n^2-n} = \sum_{n=2}^{\infty} 2^{-n} = \frac{1}{2} < 1,$$

when $\beta \geq \log 2$, we conclude that there are no positive eigenvectors at all.

Exercise 8.2. Show that $\limsup_{n\to\infty} \left(B^n_{vv}\right)^{\frac{1}{n}} = 2$.

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