Game of Life & Garden of Eden

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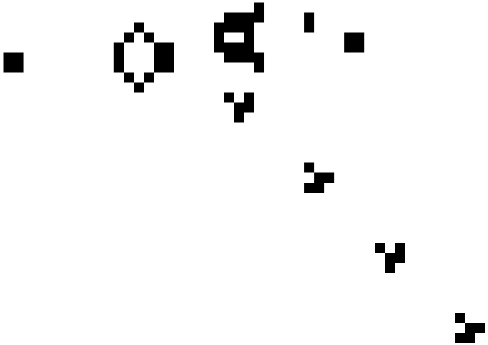
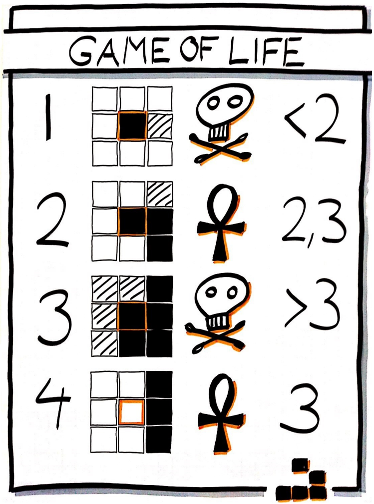
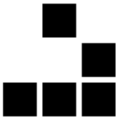


Game of Life by John Horton Conway


GAME OF LIFE
1
<2
2
2,3
3
>3
4
3

The rules of the game:

- **birth** a dead cell with three live neighbours becomes alive
- **survival** a live cell with two or three live neighbours stays alive
- **loneliness** a live cell with less than two live neighbours dies
- **overcrowding** a live cell with more than three live neighbours dies



Definition of Cellular Automata (for $G = \mathbb{Z}^d$ and $|A| < \infty$)

A map $\tau : A^G \to A^G$ is a **cellular automaton** if there is a set $S \subseteq G$ and a map $\mu : A^S \to A$ such that for all $x \in A^G$ and $g \in G$,

$$\tau(x)(g) = \mu((g^{-1}x)|_S)$$

where $(hx)(k) = x(h^{-1}k)$ for all $h, k \in G$

The Curtis–Hedlund–Lyndon Theorem (for $G = \mathbb{Z}^d$ and $|A| < \infty$)

A map $\tau : A^G \to A^G$ is a cellular automaton if and only if it is **G-equivariant** and **continuous** (w.r.t. the prodiscrete topology)

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Example – Game of Life

Put $G = \mathbb{Z}^2$, $A = \{0,1\}$ and $S = \{-1,0,1\} \times \{-1,0,1\} \subseteq G$

Define $\mu : A^S \to A$ by (for any $y \in A^S$)

$$\mu(y) = \begin{cases} 1 & \text{if } \sum_{s \in S} y(s) = 3 \\ 1 & \text{if } \sum_{s \in S} y(s) = 4 \text{ and } y((0,0)) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Define $\tau : A^G \to A^G$ as before by (for any $x \in A^G$ and $g \in G$)

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Then τ is the cellular automaton associated with the Game of Life

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A cellular automaton is

- **injective** if distinct configurations have distinct successors
- **surjective** if every configuration has a predecessor

Proposition (for $G = \mathbb{Z}^d$ and $|A| < \infty$)

A cellular automaton $\tau : A^G \to A^G$ is reversible iff it is bijective

A cellular automaton is **pre-injective** if distinct configurations, agreeing in all but a finite number of cells, have distinct successors

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Recall that a map $f : X \to Y$ is

- **injective** iff there is a map $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$
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Definition of Surjunctive Concrete Categories

A concrete category $\mathcal{C}$ is **surjunctive** if any injective endomorphism $f : X \to X$ with $X \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}(X, X)$ is surjective

An **endomorphism** is a self-map.

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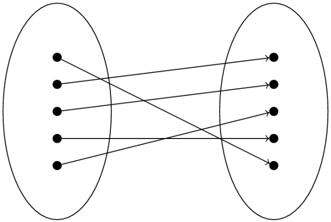
A set X is finite iff any injective map $f : X \to X$ is surjective



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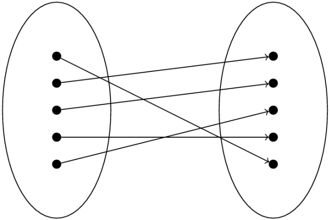

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Definition of Garden of Eden Configurations

A configuration is called a **Garden of Eden** if it has no predecessor

A cellular automaton is surjective iff it has no Garden of Eden

The Garden of Eden Theorem (for $G = \mathbb{Z}^d$ and $|A| < \infty$)

A cellular automaton $\tau : A^G \to A^G$ is pre-injective iff it is surjective

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Example – Game of Life ($G = \mathbb{Z}^2$ and $A = \{0, 1\}$)

Define configurations $x, y \in A^G$ by (for all $g \in G$)

$$x(g) = 0 \quad \text{and} \quad y(g) = \begin{cases} 1 & \text{if } g = (0,0) \\ 0 & \text{if } g \neq (0,0) \end{cases}$$

Then $\tau(x) = \tau(y) = x$ so τ is not pre-injective (nor surjective!)

A *Garden of Eden* in *Game of Life*

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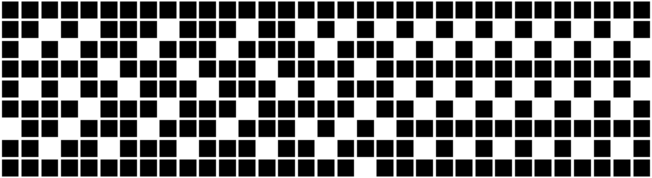

A *Garden of Eden* in *Game of Life*

Definition of Surjunctive Groups

A group G is **surjunctive** if every injective cellular automaton $\tau : A^G \to A^G$ over G with finite alphabet A is surjective

Example

$\mathbb{Z}^d$ is surjunctive by The Garden of Eden Theorem

Examples

All **finite** groups, **free** groups and **abelian** groups are surjunctive

More generally all **residually** finite groups, (residually) **amenable** groups and **sofic** groups are surjunctive

Questions

- Are all groups sofic?
- Are all groups surjunctive?
- Are all surjunctive groups sofic?

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More generally all **residually** finite groups, (residually) **amenable** groups and **sofic** groups are surjunctive

Questions

- Are there any non-sofic groups?
- Are there any non-surjunctive groups?
- Are there any non-sofic surjunctive groups?

Proposition

Any subgroup of a surjunctive group is surjunctive

A group **locally** [$\cdots$] if all its finitely generated subgroup are [$\cdots$]

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A group is surjunctive if and only if it is locally surjunctive

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Let G be a group and K a field

Kaplansky's Stable Finiteness Conjecture

The group ring $K[G]$ is stably finite

Kaplansky's Zero-Divisors Conjecture

If G is torsion-free $K[G]$ contains no zero-divisors

Kaplansky's Idempotent Conjecture

If G is torsion-free $K[G]$ contains no non-trivial idempotents

Kaplansky's Unit Conjecture

If G is torsion-free $K[G]$ contains no non-trivial units

Tullio Ceccherini-Silberstein, Michel Coornaert:
Cellular Automata and Groups, Springer-Verlag, Berlin, 2010.

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Surjunctivity and Reversibility of Cellular Automata over Concrete Categories, *Trends in Harmonic Analysis*, Springer, Milan, 2013.